(iii) $\begin{cases} k-2n-59=43 \\ k+2n+59=1 \end{cases} \Leftrightarrow k=22,\ n=-40;$

(iv) $\begin{cases} k-2n-59=-43 \\ k+2n+59=-1 \end{cases} \Leftrightarrow k=-22,\ n=-19.$

For $n=-19$ and $n=-40$, $n^2+59n+881=11^2$.

It results that $n^2+59n+881$ is a perfect square for $n=-19$ and $n=-40$.

4. Find the least positive integer n such that

$$\sqrt{3}z^{n+1}-z^n-1 = 0$$

has a complex root z with $|z|=1$.

Solved by Arkady Alt, San Jose, CA, USA.

Let $z=\cos\varphi+i\sin\varphi, \varphi\in[0,2\pi)$. Since

$$\begin{aligned}1+\cos n\varphi+i\sin n\varphi &= 2\cos^2\frac{n\varphi}{2}+2i\cos\frac{n\varphi}{2}\sin\frac{n\varphi}{2}\\ &= 2\cos\frac{n\varphi}{2}\left(\cos\frac{n\varphi}{2}+i\sin\frac{n\varphi}{2}\right)\end{aligned}$$

then

$$\begin{aligned}&\sqrt{3}z^{n+1}=1+\cos n\varphi+i\sin n\varphi\\ \Longleftrightarrow\ &\sqrt{3}z^{n+1}=2\cos\frac{n\varphi}{2}\left(\cos\frac{n\varphi}{2}+i\sin\frac{n\varphi}{2}\right)\end{aligned}$$

yields

$$\begin{aligned}&\sqrt{3}\left|z^{n+1}\right|=2\left|\cos\frac{n\varphi}{2}\left(\cos\frac{n\varphi}{2}+i\sin\frac{n\varphi}{2}\right)\right|\\ \Longleftrightarrow\ &\sqrt{3}\left|z\right|^{n+1}=2\left|\cos\frac{n\varphi}{2}\right| \Longleftrightarrow \frac{\sqrt{3}}{2}=\left|\cos\frac{n\varphi}{2}\right|\\ \Longleftrightarrow\ &\frac{\sqrt{3}}{2}=\left|\cos\frac{n\varphi}{2}\right| \Longleftrightarrow \frac{3}{2}=2\cos^2\frac{n\varphi}{2} \Longleftrightarrow \cos n\varphi=\frac{1}{2}.\end{aligned}$$

If $\cos n\varphi=\frac{1}{2}$ then $\sin n\varphi=\pm\frac{\sqrt{3}}{2}$ and $z^n=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$

Since $z^n+1=\frac{3}{2}\pm\frac{i\sqrt{3}}{2}$, $z^{n+1}=z\cdot z^n=z\left(\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)$ then

$$\begin{aligned}&\sqrt{3}z^{n+1}=z^n+1 \Longleftrightarrow \sqrt{3}z\left(\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)=\frac{3}{2}\pm\frac{i\sqrt{3}}{2}\\ \Longleftrightarrow\ &z\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)=\left(\frac{\sqrt{3}}{2}\pm\frac{1}{2}i\right)\\ \Longleftrightarrow\ &z\left(\cos\left(\pm\frac{\pi}{3}\right)+i\sin\left(\pm\frac{\pi}{3}\right)\right)=\cos\left(\pm\frac{\pi}{6}\right)+i\sin\left(\pm\frac{\pi}{6}\right)\\ \Longleftrightarrow\ &z=\cos\left(\mp\frac{\pi}{6}\right)+i\sin\left(\mp\frac{\pi}{6}\right).\end{aligned}$$

Hence, $\pm\frac{\pi}{3} = \mp\frac{n\pi}{6} + 2k\pi \iff n = \pm 12k - 2, k \in \mathbb{Z}$ and, therefore, the smallest positive integer n satisfying this equation is $n = 10$. So, a necessary condition is $n \geq 10.$

Let $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ and $n = 10$ then

$$z^{10} = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2},$$
$$z^{11} = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}i,$$
$$\text{and } \sqrt{3}z^{n+1} = \frac{3}{2} - \frac{\sqrt{3}}{2}i = 1 + \frac{1}{2} - \frac{\sqrt{3}}{2} = 1 + z^n.$$

Thus, the least positive integer n such that $\sqrt{3}z^{n+1} - z^n - 1 = 0$ has a complex root with $|z| = 1$ is $10.$

5. Let p_k denote the k^{th} prime number. Find the remainder when

$$\sum_{k=2}^{2550} p_k^{p_k^4 - 1}$$

is divided by 2550.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

First we factor 2550 into prime powers: $2550 = (255)\cdot(10) = 5\cdot 51\cdot 10 = 5\cdot 3\cdot 17\cdot 2\cdot 5$. So, we easily see that

$$2550 \;=\; 2\cdot 3\cdot 5^2\cdot 17. \tag{1}$$

First, note that $p_2 = 3$, $p_3 = 5$, and $p_7 = 17$. We will first find the congruence classes the three integers $p_2^{p_2^4-1}$, $p_3^{p_3^4-1}$, and $p_7^{p_7^4-1}$; belong to modulo 2550. We start with $p_2^{p_2^4-1} = 3^{3^4-1} = 3^{80}$.

Clearly

$$3^{80} \;\equiv\; 0 \pmod 3 \quad \text{and} \quad 3^{80} \;\equiv\; 1 \pmod 2. \tag{2}$$

By Fermat's Little Theorem, $3^{16} \equiv 1 \pmod{17}$; and so

$$3^{80} \;=\; (3^{16})^5 \;\equiv\; 1^5 \;\equiv\; 1 \pmod{17}. \tag{3}$$

Consider 3^{80} modulo $5^2 = 25$. First

$$3^8 \equiv \left(3^4\right)^2 \equiv (81)^2 \equiv (6)^2 \equiv 36 \equiv 11 \pmod{25}.$$

So that,

$$\begin{aligned} 3^{80} = (3^8)^{10} &\equiv (11)^{10} \;\equiv (11^2)^5 \;\equiv\; (121)^5 \;\equiv\; (-4)^5 \\ &\equiv\; (-4)^4\cdot(-4) \;\equiv\; (256)(-4) \;\equiv\; 6\cdot(-4) \\ &\equiv\; -24 \;\equiv\; 1 \pmod{25} \end{aligned} \tag{4}$$